Renormalization Group for Directed Sandpile Models

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Following the lead of earlier researchers, we construct a real space renormalization group map for a class of self-organized critical models. We then test a series of quantitative predictions implied by this map against direct simulations of various models. The theory passes four tests, but apparently fails a fifth. We also find that the map has an interesting nongeneric property, suggesting a deeper structure to the theory than previously appreciated. [S0031-9007(98)06840-9]

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The subject of self-organized criticality (SOC) continues to stimulate a great deal of interest. While the extent of the connection between SOC and laboratory experiments is a subject of ongoing debate [1], theoretical progress continues apace [2–8]. Particularly promising are efforts to explain the behavior of these complex dynamical systems at a fundamental level, i.e., using ideas that go beyond phenomenology or computer simulations. Typically restricted to certain subclasses of models, several intriguing approaches can be found in the literature; these include, e.g., exploiting algebraic structures [3,4], connections with singular diffusion equations [5], analysis of extremal dynamics [6], and renormalization group approaches [2,7].

The immediate inspiration for the present work is a real space renormalization group (RG) theory of SOC proposed by Pietronero, Vespignani, and Zapperi [2]. The goals of that theory are ambitious: to explain the origin of power law avalanche distributions, to calculate from first principles the critical exponents, to explain the apparent universality of these exponents, and to clarify the relationship between self-organized criticality and equilibrium critical phenomena. While this approach is dogged by the usual problems of real space RG (e.g., certain difficulties in making systematic approximations), it provides an attractive framework for understanding and explaining a number of properties exhibited by sandpile models, and does a good job of calculating the avalanche exponents for a class of models which includes the original sandpile automaton of Bak *et al.* [9].

The purpose of this Letter is to explore more fully, in the context of a specific example, the consequences implied by a real space RG approach to SOC. We consider a class of conservative sandpile models on a directed two-dimensional lattice, explicitly construct an RG map, and test five consequences predicted by quantitative analysis of the map. The constructed theory does a good job: it passes four of five tests; the lone discrepancy is rather interesting insofar as nontrivial scaling is observed even for cases where it is not predicted. Overall, we think this level of agreement is impressive, and suggest that the real space RG approach to SOC is a promising avenue of study. What is more, we find an unexpected property of the RG map which implies a deeper structure to the theory than previously appreciated.

Our choice of directed models is doubly motivated. First, exact results for the avalanche area exponent are available for a particular model within the class we consider [8]. Second, the directed nature of the dynamics makes the construction and final form of the RG map relatively simple [7], which facilitates its detailed analysis.

We begin by defining the class of dynamical models to be considered. On a two-dimensional square lattice of size $L \times L$, we define $h(i,j)$ as the integer-valued height at site (i,j). The system is perturbed by choosing at random one of the sites on the top row ($i = 1$) and incrementing its height by one unit. If the height at any site exceeds the threshold value h^*, then the site relaxes in one of three ways: (i) with probability p_R one grain is transferred to its lower-right neighbor, or (ii) with probability p_L one grain is transferred to its lower-left neighbor, or (iii) with probability $(1 - p_R - p_L)$ one grain each is transferred to both forward neighbors [Excited sites undergo *one* relaxation, regardless of their final (after relaxation) value.] The boundary conditions are periodic in the horizontal direction and open on the bottom. This class of models is a generalization of the one studied by Dhar and Ramaswamy [8], which corresponds to the special case $p_R = p_L = 0$. In what follows, we refer to this particular case as the DR model. For the DR model the area avalanche distribution exponent is known exactly, e.g., in the scaling region $P(s) \sim s^{-\tau}$ with $\tau = 4/3$ where $P(s)$ is the probability that precisely s distinct sites relax in a single avalanche. In Fig. 1 we plot (with diamonds) $P(s)$ obtained by simulating the DR model to illustrate two points. First, even for such a small system we get a recognizable scaling region (the measured exponent is 1.34). Second, for very small avalanches the data deviate from this scaling behavior—the observed point to point

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scatter is not a sampling effect: it is quite reproducible. We will return to this small scale behavior later.

We now construct the RG map, along the general lines of Ref. [2]. We coarse grain the system by considering 2×2 cells, and generate all possible relaxation sequences of an initially excited cell. Since our enumeration of the various possibilities differs somewhat from Ref. [2], it is worthwhile to go through an example in detail. Figure 2a shows one relaxation sequence, which ultimately results in spillage of a single grain to the right neighbor cell. At each step in the sequence we allow one of four events to occur: absorption (no grains spilled) with probability x_0 [10]; one grain spilled to the lower right (left) with probability $u_0(v_0)$; or two grains spilled, one each to the right and left with probability z_0. Treating successive events as independent, the path shown in Fig. 2a occurs with probability $z_0 x_0 u_0$.

Generating all possible sequences, we find that there are a total of nine different ways a given cell can affect its neighbors (see Fig. 2b). For each there is a probability amplitude; the sequence in Fig. 2a contributes to the amplitude P. After running through all possible paths, we get expressions for all eight amplitudes:

$$P = u^2 + 2u^2 v + 2uxz + 2u^2 xz + uvxz + 2u^2 vxz + x^2 z^2 + ux^2 z^2, \quad (1)$$

$$Q = u^2 z + u^3 z + u^3 vz + uxz^2 + 2u^2 xz^2,$$

$$R = v^2 + 2uv^2 + 2vxz + uvxz + 2v^2 xz + 2uv^2 xz + x^2 z^2 + vx^2 z^2,$$

$$S = v^2 z + v^3 z + uv^3 z + vxz^2 + 2v^2 xz^2,$$

$$Y = uz^2 + 2u^2 z^2 + uvz^2 + 5u^2 vz^2 + xz^3 + 4uxz^3,$$

$$W = vz^2 + uvz^2 + 2v^2 z^2 + 5uv^2 z^2 + xz^3 + 4vxz^3,$$

$$F = 5uvz + u^2 vz + uv^2 z + 2u^2 v^2 z + 3uxz^2 + 3vxz^2 + 6uvxz^2 + x^2 z^3,$$

$$G = uz^3 + vz^3 + 7uvz^3 + 2xz^4,$$

where we have suppressed the subscript zero on x, u, v, and z.

The next step is to determine the appropriate coarse grained probabilities for the original four events, which we denote by the amplitudes x_1, u_1, v_1, and z_1. The simplest way to do this is to take:

$$x_1 = (F + Q + S + 2Y + 2W + 3G)/N, \quad (2)$$

$$u_1 = (P + Q)/N,$$

$$v_1 = (R + S)/N,$$

$$z_1 = (F + Y + W + G)/N,$$

where $N = x_1 + u_1 + v_1 + z_1$ normalizes the probabilities. The last three expressions are self-evident (compare Fig. 2b); the expression for x_1 ensures that the coarse grained relaxation rule is conservative (that is, on the average as many grains go into a cell as spill out).

Taken together, (1) and (2) define the RG map on the four-dimensional space (x_n, u_n, v_n, z_n). [A significant advantage of these directed models is the compactness of

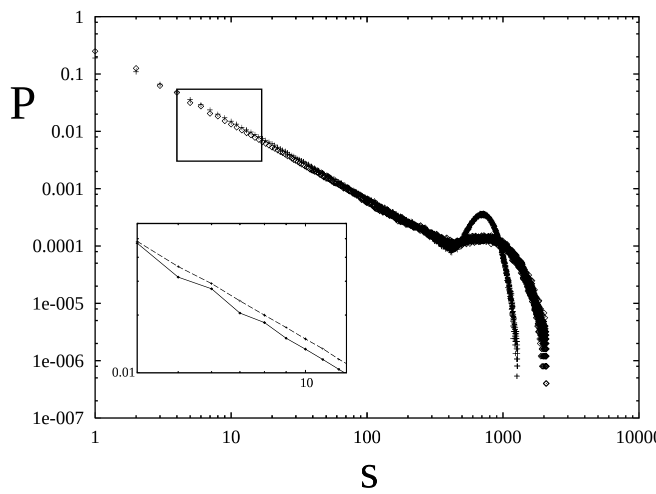

FIG. 1. Avalanche distribution for the DR model (diamonds) and for the model corresponding to the fixed point of the RG map (crosses). The inset highlights the nonpower law behavior of the DR model for small avalanches, and the corresponding power law scaling at small length scales for the fixed point model.

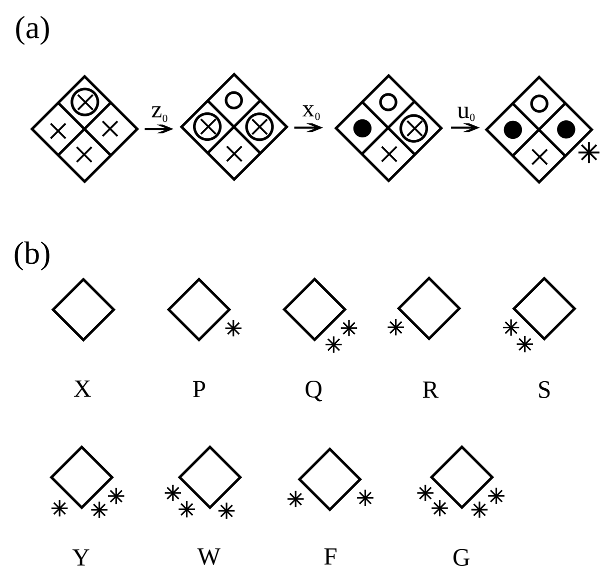

FIG. 2. (a) Example of the renormalization scheme. The cross denotes a generic site and the filled circle represents a critical (but stable) site; the additional ring denotes a supercritical site which must then relax. The open circles denote sites which have relaxed, and the asterisks denote the spillage of sand into the neighboring diamonds. Shown is the dynamical path which generates a P event (see text). (b) The nine 2×2 cell relaxation events.

the enumeration process and of the resulting RG map: in generating Eq. (1) there are only 106 distinct paths to consider, rather than many thousands of paths for the nondirected models [2]; the resulting map in the latter case has hundreds of terms. The relative simplicity of the present map makes the subsequent analysis considerably easier.] We are now ready to explore and test the various consequences of the RG map.

The first result comes from iterating the RG map starting from the initial condition corresponding to the DR model $(x_0, u_0, v_0, z_0) = (\frac{1}{2}, 0, 0, \frac{1}{2})$. Under successive iterations of the RG map, the induced orbit is attracted to the fixed point $(0.373, 0.206, 0.206, 0.215)$. Following Ref. [2] let $K(s)$ be the conditional probability that an avalanche fails to propagate beyond scale $s + 1$ given that it propagates to scale s. This is related to the toppling amplitudes via $K = (zx^2 + ux + vx + 2uvx + zx^2v + zx^2u)/(1 - x)$, where the numerator is the probability of not propagating out of a composite cell and the denominator is the probability that the avalanche propagates beyond the previous scale. In general, successive values of K along the orbit allow one to reconstruct the avalanche area distribution scale by scale. Near the fixed point, K approaches a constant K^* and the avalanche distribution approaches a power law with $K^* = 1 - 2^{2(1-\tau)}$. Using the fixed point values yields $\tau = 1.325$, in excellent agreement with the exact value $4/3$.

The second consequence we can test concerns the universality of this exponent. For example, we can simulate a model with a different relaxation rule than DR, say where the three different relaxation events are chosen with equal probability. In our space of models, this corresponds to the initial condition $(x_0, u_0, v_0, z_0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. By iterating the RG map, we find the resulting orbit is attracted to the same fixed point as before; consequently, in the scaling regime we expect to recover the same exponent. Figure 3 shows the avalanche area distribution resulting from numerical simulations using this "mixed" rule. As expected, a scaling region with exponent 1.33 is obtained.

A third consequence of the theory is that *the scaling regime should extend all the way down to the smallest length scales* for the particular microscopic model corresponding to the fixed point, namely $(x_0, u_0, v_0, z_0) = (0.373, 0.206, 0.206, 0.215)$. That is, we simulate a model where for each relaxation event, we choose one of the three rules in the ratio 0.206:0.206:0.215 for spilling one grain to the left, one to the right, and one each to the left and right, respectively. Figure 1 shows the results of these simulations together with those for the DR model for comparison. Indeed, the fixed point model shows excellent scaling behavior at even the smallest scales.

We can learn still more by studying in detail the RG map in the vicinity of the fixed point, and this leads us to another prediction, and also our first surprise. Since

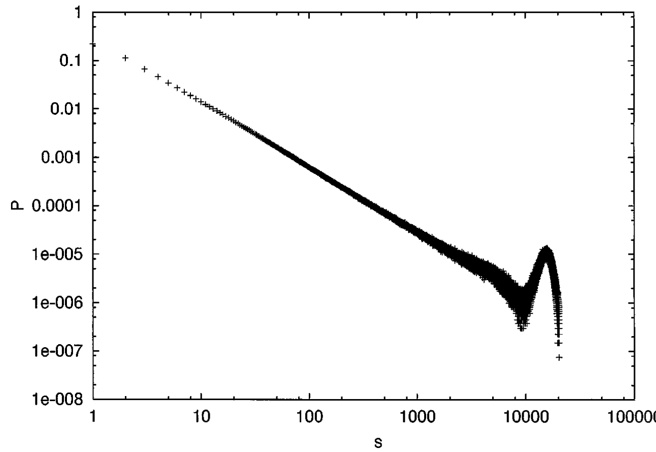

FIG. 3. Avalanche distribution for the rule (0.25, 0.25, 0.25, and 0.25).

we have an explicit representation of the map, we can calculate the 4×4 Jacobian matrix analytically. The four eigenvalues corresponding to the above mentioned fixed point are found to be $0.000, 0.001, 0.499$, and 1.421. The new prediction is that, since one of the eigenvalues has a magnitude bigger than unity, the fixed point is unstable. The surprise is that two of the eigenvalues are (nearly) zero. Let's consider each of these two facts in turn.

Consider first the unstable eigenvalue. It existence means that there are models which fall outside the DR universality class. (Indeed, it means that a typical model falls outside of it.) We can identify the DR universality class by considering the (left) eigenvector corresponding to the unstable eigenvalue. It is $(0, 1, -1, 0)$, and is therefore orthogonal to the subspace $u_0 = v_0$. Figure 4 shows a sketch of the phase space flow in the u-v plane. The fixed point lies on the line $u = v$, and its instability implies that models which break the left-right symmetry should fall outside of the DR universality class. We tested this by simulating the model corresponding to $(x_0, u_0, v_0, z_0) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$, and found a scaling region with an avalanche exponent $\tau = 1.29$, which is close to but clearly distinguishable from the DR exponent $\tau = 4/3$.

While these last data verify one prediction of the theory, it contradicts another one: namely, we shouldn't expect to see scaling behavior at all. Starting from this initial condition, successive iterates of the RG map are attracted to the fixed point $(0, 0, 1, 0)$ (see Fig. 4), corresponding to $K^* = 0$, which implies that $P(s)$ decays more slowly than $1/s$. Clearly, since this cannot hold in the infinite size limit, this situation forces us to consider what happens at the boundaries. It is therefore possible that boundary effects are dominating the data. Another possibility lies in the singularity of the avalanche distribution for any rule with $x = 0$. These rules have all sites critical (since there is zero probability of absorbing a grain), and the pile is always in the minimally stable state. Avalanches are then of one size only—the system

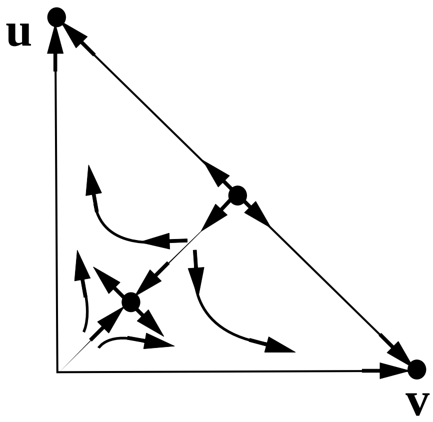

FIG. 4. Phase space flow projected onto the u-v plane, $u + v \leq 1$. The nontrivial fixed point attracts all points with $u = v$ symmetry, but is unstable with respect to breaking this symmetry. The fixed points $(1,0)$ and $(0,1)$ are stable, and $(\frac{1}{2},\frac{1}{2})$ is unstable.

size—and the distribution is singular. Regardless, the existing theoretical framework does not include such considerations.

Finally, we turn our attention to the (nearly) double-zero eigenvalues of the DR fixed point. This is a subtle issue, but nevertheless a provocative one. We know from the general theory of iterative maps that the appearance of a zero eigenvalue is special: it often signals that the map has some special (i.e., nongeneric) property. In fact, we can prove that one eigenvalue must be exactly zero, since the RG map conserves probability, i.e., $x_n + u_n + v_n + z_n = 1$. This is a special property viewed from the general perspective of iterative maps, although, of course, it is a necessary property due to the physical meaning of the RG theory. The real puzzle concerns the other (nearly) zero eigenvalue. It could be that it just happens to be very very small, but it is suspiciously close to zero. (Recall that our RG construction is not exact, so 0.001 may well indicate zero for the "true" map.) If the eigenvalue is indeed constrained to be zero, what is the underlying structural reason? We haven't found an answer, and leave it as an open question. (One can readily check that it does *not* follow from local conservation of sand.)

In summary, we have constructed a real space RG map for a class of directed sandpile automata, and tested several of its predictions. The theory accurately predicts the following: (i) the value of the exponent in the case where it is known exactly, (ii) its universality for a model with a different but symmetric relaxation rule, (iii) that the particular model corresponding to the nontrivial fixed point shows power law scaling down to the smallest length scales, and (iv) that a model which breaks the left-right symmetry in the relaxation rule falls outside of the DR universality class. However, in this last instance, simulations show nontrivial scaling behavior which is not expected from the RG map.

The lone discrepancy shows that at least some modification is needed, although it is unclear whether the flaw resides in our particular construction of the transformation or is more fundamental. There is also a question whether this level of agreement is peculiar to certain types of models (e.g., directed models have the special property that there are no back avalanches) and will grow worse when extended to include a more general class of models. Taken as a whole, however, the level of agreement is quite good, suggesting that a real space RG approach is a promising avenue for a fundamental understanding of self-organized criticality.

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