

Correlation resonance generated by coupled enzymatic processing: Supplementary information

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Throughout this supplement, N and Q are assumed to have their steady-state distributions (with either $K = 0$ or $K > 0$). In the first section, we extend the results on the moments of N given for $L = 1$ in the main text to the case $L > 1$, from which expressions for the moments and correlations for Q follow immediately. We treat the case $K = 0$ first and then the case $K > 0$. In the second section, we give a proof for $L = 1$ that the correlations of Q are non-negative. (We believe that this property also holds for $L > 1$, but we do not have a proof of this.) In the third section, to accompany our numerical findings concerning peak correlation, we present some analysis of critical points of the correlations of Q when considered as functions of the total production rate Λ and the positive probabilities p_i , $i = 1, \dots, m$. In the fourth section, we describe the numerical procedure used for finding the value and location of the peak correlation when the number of species m is two (see Fig. 3 of the main text). In the final section, we report on the results of stochastic simulations performed for biochemical reactions associated with variants of the stochastic model where we allow the degradation rates to depend on the species being degraded or we allow copies of the enzyme to be produced randomly and to be removed by dilution. Although such variations yield a model beyond the range of our exact analysis, sample simulations suggest that our qualitative findings regarding correlations are robust to such variations.

1 Moments of N for $L > 1$

Throughout, $\Lambda = \sum_{i=1}^m \lambda_i$ and $p_i = \lambda_i/\Lambda$ for $i = 1, \dots, m$.

1.1 The case $K=0$.

For this, let

$$\zeta = \frac{\Lambda}{\mu + \gamma}, \quad \beta = \frac{L\mu}{\gamma} + L, \quad \delta = \frac{\Lambda}{\gamma}.$$

To obtain an expression for the moment generating function of N , we note that

$$\begin{aligned} \prod_{\ell=1}^n \phi(\ell) &= \begin{cases} (\mu + \gamma)^n n! & \text{for } n \leq L, \\ (\mu + \gamma)^L L! \prod_{\ell=L+1}^n (L\mu + \ell\gamma) & \text{for } n > L, \end{cases} \\ &= \begin{cases} (\mu + \gamma)^n n! & \text{for } n \leq L, \\ (\mu + \gamma)^L L! \gamma^{n-L} (\beta + 1)_{n-L} & \text{for } n > L, \end{cases} \end{aligned} \quad (1)$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the rising factorial. Then the normalizing constant c satisfies

$$\begin{aligned} c^{-1} &= \sum_{n=0}^{L-1} \frac{\zeta^n}{n!} + \frac{\zeta^L}{L!} \sum_{n=L}^{\infty} \frac{\delta^{n-L}}{(\beta + 1)_{n-L}} \\ &= G_L(\zeta) + H_L(\zeta, 1, \beta + 1, \delta) \end{aligned} \quad (2)$$

where

$$\begin{aligned} G_L(w) &= \sum_{n=0}^{L-1} \frac{w^n}{n!} \\ H_L(w, x, y, z) &= \frac{w^L}{L!} M(x, y, z) \end{aligned}$$

and M is the confluent hypergeometric function as in the main text. The function G_L can also be written in terms of the incomplete Gamma function:

$$G_L(w) = \frac{e^w \Gamma(L, w)}{(L-1)!},$$

where $\Gamma(L, w) = \int_w^{\infty} t^{L-1} e^{-t} dt$. Similarly, the moment generating function for N is given by:

$$E[e^{uN}] = c \sum_{n=0}^{\infty} \frac{(e^u \Lambda)^n}{\prod_{\ell=1}^n \phi(\ell)} = c (G_L(e^u \zeta) + H_L(e^u \zeta, 1, \beta + 1, e^u \delta)), \quad u \in \mathbb{R}.$$

By differentiating this expression and using the identity:

$$\frac{d}{dz} M(x, y, z) = \frac{x}{y} M(x+1, y+1, z),$$

we obtain:

$$\begin{aligned} E[N] &= \frac{d}{du} E[e^{uN}] \Big|_{u=0} \\ &= c \left(e^u \zeta G'_L(e^u \zeta) + e^u \zeta H_L^{(1)}(e^u \zeta, 1, \beta + 1, e^u \delta) + e^u \delta H_L^{(4)}(e^u \zeta, 1, \beta + 1, e^u \delta) \right) \Big|_{u=0} \\ &= c \left(\zeta G_{L-1}(\zeta) + \zeta H_{L-1}(\zeta, 1, \beta + 1, \delta) + \frac{\delta}{\beta + 1} H_L(\zeta, 2, \beta + 2, \delta) \right), \end{aligned}$$

where c is given by Eq. 2, $G'_L = G_{L-1}$ is the first derivative of G_L , $H_L^{(1)} = H_{L-1}$ is the first partial derivative of H_L with respect to its first argument, and $H_L^{(4)}(w, x, y, z) = \frac{x}{y} H_L(w, x+1, y+1, z)$ is the first partial derivative of $H_L(w, x, y, z)$ with respect to its fourth argument. Similarly,

$$\begin{aligned}
E[N^2] &= c \frac{d}{du} \left(e^u \zeta G_{L-1}(e^u \zeta) + e^u \zeta H_{L-1}(e^u \zeta, 1, \beta + 1, e^u \delta) \right. \\
&\quad \left. + \frac{e^u \delta}{\beta + 1} H_L(e^u \zeta, 2, \beta + 2, e^u \delta) \right) \Big|_{u=0} \\
&= E[N] + c \left((e^u \zeta)^2 G'_{L-1}(e^u \zeta) + (e^u \zeta)^2 H_{L-1}^{(1)}(e^u \zeta, 1, \beta + 1, e^u \delta) \right. \\
&\quad \left. + e^{2u} \zeta \delta H_{L-1}^{(4)}(e^u \zeta, 1, \beta + 1, e^u \delta) + \frac{e^{2u} \zeta \delta}{\beta + 1} H_L^{(1)}(e^u \zeta, 2, \beta + 2, e^u \delta) \right. \\
&\quad \left. + \frac{(e^u \delta)^2}{\beta + 1} H_L^{(4)}(e^u \zeta, 2, \beta + 2, e^u \delta) \right) \Big|_{u=0} \\
&= E[N] + c \left(\zeta^2 G_{L-2}(\zeta) + \zeta^2 H_{L-2}(\zeta, 1, \beta + 1, \delta) \right. \\
&\quad \left. + \frac{2\zeta \delta}{\beta + 1} H_{L-1}(\zeta, 2, \beta + 2, \delta) + \frac{2\delta^2}{(\beta + 1)(\beta + 2)} H_L(\zeta, 3, \beta + 3, \delta) \right).
\end{aligned}$$

By substituting these formulas for c , $E[N]$ and $E[N^2]$ into Eqs. 16–19 and Eq. 8 in the main text, one can obtain formulas for the means, variances, covariances and correlations associated with Q . When $L = 1$, these formulas reduce, after some manipulation, to those given in the main text. Recall that to simplify our model, we assumed that protein molecules bound to the enzyme could be removed by dilution. Analogous formulas that correspond to the same model, but where molecules bound to the enzyme are not subject to dilution, can be obtained simply by replacing μ by $\mu - \gamma$ in the above formulas.

1.2 The case $K > 0$.

For this case, we set

$$\zeta = \frac{\Lambda}{\mu + \gamma}, \quad \alpha = K + L, \quad \beta = \frac{L\mu}{\gamma} + \alpha, \quad \delta = \frac{\Lambda}{\gamma}, \quad \kappa = \frac{K\gamma}{\mu + \gamma}.$$

In this case, we have for $n \leq L$,

$$\prod_{\ell=1}^n \phi(\ell) = \prod_{\ell=1}^n \left(\frac{\ell^2 \mu}{K + \ell} + \ell \gamma \right) = (\mu + \gamma)^n n! \frac{(\kappa + 1)_n}{(K + 1)_n}$$

and for $n > L$,

$$\begin{aligned}
\prod_{\ell=1}^n \phi(\ell) &= (\mu + \gamma)^L L! \frac{(\kappa + 1)_L}{(K + 1)_L} \prod_{\ell=L+1}^n \left(\frac{L\ell\mu}{K + \ell} + \ell \gamma \right) \\
&= (\mu + \gamma)^L L! \frac{(\kappa + 1)_L}{(K + 1)_L} \frac{\gamma^{n-L} (L + 1)_{n-L} (\beta + 1)_{n-L}}{(\alpha + 1)_{n-L}}.
\end{aligned}$$

Note that these formulas reduce to those in Eq. 1 on setting $K = 0$. The normalizing constant c then satisfies

$$\begin{aligned} c^{-1} &= \sum_{n=0}^{L-1} \frac{\zeta^n (K+1)_n}{n! (\kappa+1)_n} + \frac{\zeta^L (K+1)_L}{L! (\kappa+1)_L} \sum_{n=L}^{\infty} \frac{(\alpha+1)_{n-L} \delta^{n-L}}{(L+1)_{n-L} (\beta+1)_{n-L}} \\ &= \tilde{G}_L(K+1, \kappa+1, \zeta) + \tilde{H}_L(K+1, \kappa+1, \zeta; 1, \alpha+1, L+1, \beta+1, \delta) \end{aligned}$$

where

$$\begin{aligned} \tilde{G}_L(u, v, w) &= \sum_{n=0}^{L-1} \frac{w^n (u)_n}{n! (v)_n} \\ \tilde{H}_L(u, v, w; r, x, s, y, z) &= \frac{w^L (u)_L}{L! (v)_L} {}_2F_2(r, x; s, y; z) \end{aligned}$$

and ${}_2F_2$ is the generalized hypergeometric function [1]:

$${}_2F_2(r, x; s, y; z) = \sum_{n=0}^{\infty} \frac{(r)_n (x)_n z^n}{(s)_n (y)_n n!}. \quad (3)$$

It is straightforward to prove the identity

$$\tilde{G}_L(u, v, w) = M(u, v, w) - \frac{w^L (u)_L}{L! (v)_L} {}_2F_2(1, u+L; L+1, v+L; w) \quad (4)$$

which can be used to compute $\tilde{G}_L(u, v, w)$ efficiently, e.g., when L is large. The moment generating function for N is given for $u \in \mathbb{R}$ by

$$E[e^{uN}] = c \left(\tilde{G}_L(K+1, \kappa+1, e^u \zeta) + \tilde{H}_L(K+1, \kappa+1, e^u \zeta; 1, \alpha+1, L+1, \beta+1, e^u \delta) \right).$$

We note that

$$\begin{aligned} \tilde{G}_L^{(3)}(u, v, w) &= \frac{\partial \tilde{G}_L}{\partial w}(u, v, w) = \frac{u}{v} \tilde{G}_{L-1}(u+1, v+1, w) \\ \tilde{H}_L^{(3)}(u, v, w; r, x, s, y, z) &= \frac{\partial \tilde{H}_L}{\partial w}(u, v, w; r, x, s, y, z) = \frac{u}{v} \tilde{H}_{L-1}(u+1, v+1, w; r, x, s, y, z) \\ \tilde{H}_L^{(8)}(u, v, w; r, x, s, y, z) &= \frac{\partial \tilde{H}_L}{\partial z}(u, v, w; r, x, s, y, z) \\ &= \frac{rx}{sy} \tilde{H}_L(u, v, w; r+1, x+1, s+1, y+1, z). \end{aligned}$$

Thus, by differentiating the moment generating function we obtain

$$\begin{aligned}
E[N] &= \frac{d}{du} E[e^{uN}]|_{u=0} \\
&= c \left(e^u \zeta \tilde{G}_L^{(3)}(K+1, \kappa+1, e^u \zeta) \right. \\
&\quad \left. + e^u \zeta \tilde{H}_L^{(3)}(K+1, \kappa+1, e^u \zeta; 1, \alpha+1, L+1, \beta+1, e^u \delta) \right. \\
&\quad \left. + e^u \delta \tilde{H}_L^{(8)}(K+1, \kappa+1, e^u \zeta; 1, \alpha+1, L+1, \beta+1, e^u \delta) \right) \Big|_{u=0} \\
&= c \left(\zeta \frac{K+1}{\kappa+1} \tilde{G}_{L-1}(K+2, \kappa+2, \zeta) \right. \\
&\quad \left. + \zeta \frac{K+1}{\kappa+1} \tilde{H}_{L-1}(K+2, \kappa+2, \zeta; 1, \alpha+1, L+1, \beta+1, \delta) \right. \\
&\quad \left. + \delta \frac{\alpha+1}{(L+1)(\beta+1)} \tilde{H}_L(K+1, \kappa+1, \zeta; 2, \alpha+2, L+2, \beta+2, \delta) \right). \\
\\
E[N^2] &= c \frac{d}{du} \left(e^u \zeta \frac{K+1}{\kappa+1} \tilde{G}_{L-1}(K+2, \kappa+2, e^u \zeta) \right. \\
&\quad \left. + e^u \zeta \frac{K+1}{\kappa+1} \tilde{H}_{L-1}(K+2, \kappa+2, e^u \zeta; 1, \alpha+1, L+1, \beta+1, e^u \delta) \right. \\
&\quad \left. + e^u \delta \frac{\alpha+1}{(L+1)(\beta+1)} \tilde{H}_L(K+1, \kappa+1, e^u \zeta; 2, \alpha+2, L+2, \beta+2, e^u \delta) \right) \Big|_{u=0} \\
&= E[N] + \\
&\quad c \left(\zeta^2 \frac{(K+1)(K+2)}{(\kappa+1)(\kappa+2)} \tilde{G}_{L-2}(K+3, \kappa+3, \zeta) \right. \\
&\quad \left. + \zeta^2 \frac{(K+1)(K+2)}{(\kappa+1)(\kappa+2)} \tilde{H}_{L-2}(K+3, \kappa+3, \zeta; 1, \alpha+1, L+1, \beta+1, \delta) \right. \\
&\quad \left. + 2\zeta \delta \frac{K+1}{\kappa+1} \frac{\alpha+1}{(L+1)(\beta+1)} \tilde{H}_{L-1}(K+2, \kappa+2, \zeta; 2, \alpha+2, L+2, \beta+2, \delta) \right. \\
&\quad \left. + \delta^2 \frac{2(\alpha+1)(\alpha+2)}{(L+1)(L+2)(\beta+1)(\beta+2)} \tilde{H}_L(K+1, \kappa+1, \zeta; 3, \alpha+3, L+3, \beta+3, \delta) \right).
\end{aligned}$$

By substituting these formulas for c , $E[N]$ and $E[N^2]$ into Eqs. 16–19 and Eq. 8 in the main text, one can obtain formulas for the means, variances, covariances and correlations associated with Q . One could consider a model where molecules bound to the enzyme are not subject to dilution. The only change would be to replace ϕ with

$$\phi(n) = \begin{cases} \frac{n^2 \mu}{K+n} & \text{for } n \leq L, \\ \frac{Ln\mu}{K+n} + (n-L)\gamma & \text{for } n > L. \end{cases} \quad (5)$$

Unfortunately, the product $\prod_{\ell=1}^n \phi(\ell)$ does not seem to simplify easily in this case and evaluations of the moments of N will involve rather complicated expressions.

2 Proof that correlations for Q are non-negative for $L = 1$

For $L = 1$, we show that the correlations r_{ij} for $j \neq i$ are non-negative. (The correlation r_{ii} is always one.) From the expressions given in the main text for the correlations in terms of moments of N , we see that it suffices to prove that $Var(N) - E[N]$ is non-negative. For the proof, we use the formulas developed in the main text for $K > 0$, which are also valid for $K = 0$. For α, β, δ given by:

$$\alpha = K + 1, \quad \beta = \frac{\mu}{\gamma} + \alpha, \quad \delta = \frac{\Lambda}{\gamma},$$

we have

$$\begin{aligned} Var(N) - E[N] &= E[N^2] - (E[N])^2 - E[N] \\ &= \frac{\alpha(\alpha + 1)\delta^2 M(\alpha + 2, \beta + 2, \delta)}{\beta(\beta + 1)M(\alpha, \beta, \delta)} - \left(\frac{\alpha\delta M(\alpha + 1, \beta + 1, \delta)}{\beta M(\alpha, \beta, \delta)} \right)^2 \\ &= \left(\frac{\delta}{\beta M(\alpha, \beta, \delta)} \right)^2 \left(\frac{\alpha(\alpha + 1)}{1 + \frac{1}{\beta}} M(\alpha + 2, \beta + 2, \delta) M(\alpha, \beta, \delta) - (\alpha M(\alpha + 1, \beta + 1, \delta))^2 \right). \end{aligned} \quad (6)$$

Now, using the Cauchy-Schwarz inequality, for the last term above we have

$$\begin{aligned} (\alpha M(\alpha + 1, \beta + 1, \delta))^2 &= \alpha^2 \left(\sum_{n=0}^{\infty} \frac{(\alpha + 1)_n \delta^n}{(\beta + 1)_n n!} \right)^2 \\ &\leq \alpha^2 \left(\sum_{n=0}^{\infty} \frac{(\alpha + 1)(\alpha + 2)_n}{\alpha(1 + \frac{1}{\beta})} \frac{\delta^n}{(\beta + 2)_n n!} \right) \left(\sum_{n=0}^{\infty} \frac{(\alpha)_n \delta^n}{(\beta)_n n!} \right), \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \frac{((\alpha + 1)_n)^2 (\beta)_n (\beta + 2)_n}{(\alpha)_n ((\beta + 1)_n)^2} &= \frac{(\alpha + 1)_n (\alpha + n) \beta (\beta + n + 1)}{\alpha (\beta + 1) (\beta + n)} \\ &= \frac{(\alpha + 1)_n (\alpha + n + \frac{\alpha + n}{\beta + n})}{\alpha (1 + \frac{1}{\beta})} \\ &\leq \frac{(\alpha + 1)_n (\alpha + n + 1)}{\alpha (1 + \frac{1}{\beta})} \\ &= \frac{(\alpha + 1)(\alpha + 2)_n}{\alpha (1 + \frac{1}{\beta})}, \end{aligned}$$

and we used the fact that $\beta \geq \alpha$ for the inequality above. Thus,

$$(\alpha M(\alpha + 1, \beta + 1, \delta))^2 \leq \frac{\alpha(\alpha + 1)}{1 + \frac{1}{\beta}} M(\alpha + 2, \beta + 2, \delta) M(\alpha, \beta, \delta)$$

and on substituting this into Eq. 6, we obtain that $Var(N) - E[N] \geq 0$, as desired.

3 Critical points of the correlations

In terms of parameter dependence, we first note that the collection of probabilities $p_i = \lambda_i/\Lambda$, $i = 1, \dots, m$, can be varied independently of $\Lambda = \sum_{i=1}^m \lambda_i$. Furthermore, Λ and the p_i 's can be varied independently of L, μ, γ and K .

Assuming there is a unique critical point for the correlation r_{ij} for $j \neq i$ fixed, when considered as a function of Λ for each fixed set of values for the other parameters $L, \mu, \gamma, K, p_1, \dots, p_m$, we now show that the value $\Lambda = \Lambda_{max}$ where this occurs does not depend on the positive probabilities $p_i, i = 1, \dots, m$. (Our numerical explorations support the hypothesis of a unique critical point and suggest that the correlation has a unique maximum there.)

From Eqs. 8–9 of the main text and the fact that the steady-state distribution for N does not depend on the p_i 's, the correlation r_{ij} for $j \neq i$ has the form

$$r_{ij} = \frac{F(z) - 1}{\sqrt{(F(z) - 1 + 1/p_i)(F(z) - 1 + 1/p_j)}} \quad (7)$$

where

$$F(z) = \frac{Var(N)}{E[N]} \quad (8)$$

and $z = (\Lambda, L, \mu, \gamma, K)$ is independent of $p_i = \lambda_i/\Lambda$, $i = 1, \dots, m$. Here $F(z) - 1 + 1/p_i$ is strictly positive for each i . It is straightforward to show that

$$\frac{\partial r_{ij}}{\partial \Lambda} = \frac{\partial F}{\partial \Lambda} \frac{(F(z) - 1 + 1/p_i)/p_j + (F(z) - 1 + 1/p_j)/p_i}{2(F(z) - 1 + 1/p_i)^{3/2}(F(z) - 1 + 1/p_j)^{3/2}}. \quad (9)$$

All terms in Eq. 9 other than $\partial F/\partial \Lambda$ are positive. Thus the critical point (where $\partial r_{ij}/\partial \Lambda = 0$), must be the unique solution of $\partial F/\partial \Lambda = 0$. Since F does not depend on the p_i 's, it follows that this value Λ_{max} does not depend on the p_i 's.

Next, we fix $\Lambda, L, \mu, \gamma, K, j \neq i, \{p_k, k \neq i, j\}$ and consider the correlation r_{ij} as a function of p_i , where $p_j = p - p_i$ for $p = 1 - \sum_{k \neq i, j} p_k$. We assume that the correlation is strictly positive and has a unique critical point (which corresponds to a maximum for the correlation). The assumption that r_{ij} is positive implies that $F(z) - 1 > 0$. Then

$$\frac{\partial r_{ij}}{\partial p_i} = (F(z) - 1) \frac{p_j - p_i}{p_i p_j} \frac{F(z)/p_i + F(z)/p_j + (1 - p)/p_i p_j}{2(F(z) - 1 + 1/p_i)^{3/2}(F(z) - 1 + 1/p_j)^{3/2}}. \quad (10)$$

All terms in Eq. 10 other than $p_j - p_i$ are positive and so the critical point occurs where $p_i = p_j = p/2$. It follows that for two species ($m = 2$), the assumed maximum of the correlation for fixed Λ but variable p_1 occurs at the symmetric point $p_1 = p_2 = 1/2$.

4 Numerical method for finding the value and location of peak correlation

For $m = 2$, using the analytic expression for the correlation r_{12} , the ‘‘Maximize’’ routine in Maple 11 (Waterloo Maple, Inc.) was used to numerically find the maximum value of the correlation and the value of the argument $\Lambda = \Lambda_{max}$ where the maximum occurred for Fig. 3 of the main text. Bounds for the search were $0 \leq \Lambda \leq 1000$.

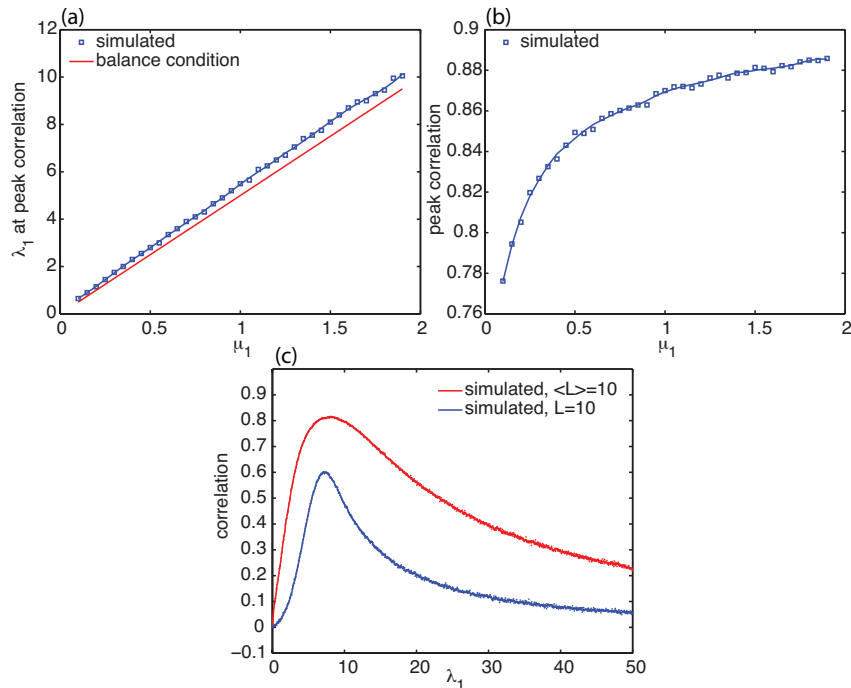


Figure 1: Results of simulation for variants of our stochastic model. (a) For two protein species ($m = 2$), different degradation rates, μ_1 and μ_2 , were allowed. For given μ_1 and fixed $\mu_2 = 1$, the steady-state correlation between the numbers of each type of protein was estimated by stochastic simulation of the biochemical reactions for a range of λ_1 (simulation time of 6×10^5). As a function of λ_1 , the correlation was again observed to have a peak; the value of λ_1 at the peak was found numerically for the correlation data filtered through a Savitzky-Golay smoothing filter. Quantitatively similar results arise if no smoothing filter is used. A plot of the value of λ_1 at the peak as a function of μ_1 is shown in (a). Other parameters used were $L = 10$, $\lambda_2 = 5$, $\gamma = 0.01$, and $\eta = 10^8$. The blue line through the data is a trend line. We find that the value of λ_1 where the peak correlation occurs is close to the balance point (red line) which satisfies the “heavy traffic condition” (see Eq. 14). (b) Similar to (a), but showing the value of the peak correlation as a function of μ_1 . The value of the peak correlation does not appear to be very sensitive to varying μ_1 . (c) With a common degradation rate for the two species, the number of copies of the enzyme is now allowed either to fluctuate around the steady-state mean value $\nu/\gamma = 10$ (red points) due to random production and dilution, or to remain fixed at $L = 10$ (blue points). Other parameters used were $\mu_1 = \mu_2 = 1$, $\lambda_2 = 5$, $\gamma = 0.1$, and $\eta = 10^8$. As λ_1 is scanned, the correlation (estimated by stochastic simulation of the biochemical reactions for a simulation time of 6×10^5) for the model with a fluctuating number of copies of the enzyme tends to be higher than the correlation for the model with a fixed number of copies of the enzyme. However, a peak in correlation is still observed in both cases near the balance point.

5 Simulation results for variants of the stochastic model

Our exact analysis of the stochastic model presented in the main text used the assumptions of a common degradation rate, μ , and a fixed number of copies, L , of the enzyme. In Fig. 1, we display results from some stochastic simulations of biochemical reactions associated with a generalization of this model where (i) the degradation rate is allowed to depend on the species being degraded, or (ii) the number of copies of the enzyme L is allowed to vary randomly by permitting production of enzyme at a constant rate and removal of enzyme by dilution. These generalizations correspond to the following changes in the system of biochemical reactions given in Eqs. 1–5 of the main text: (i) allow μ in Eq. 3 to depend on i , or (ii) replace Eq. 4 with the following set of equations:



Simulations were performed for the case of two species ($m = 2$). For small dilution rate, γ , we again find that with all parameters fixed except λ_1 , the correlation as a function of λ_1 has a peak near a balance point which is given by the heavy traffic condition:

$$\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = L. \quad (14)$$

The balance condition can be interpreted as follows. If we regard molecules of type i as bringing an amount of work (measured in units of average processing time) of size $1/\mu_i$ to the system, and we regard the enzymatic processors as working at unit rate, then at balance the rate at which “work” is arriving to the system is equal to the maximum rate at which the machinery can process that work (which equals the number of enzymatic processors in the system). The results of our sample simulations suggest that our qualitative findings regarding correlation are robust to the aforementioned variants in the stochastic model.

References

- [1] Bailey WN (1935) *Generalized Hypergeometric Series* (Cambridge University Press, Cambridge, U.K.).